Speaking about transitive frames in propositional languages(New Aspects in Non-Classical Logics and Their Kripke Semantics)

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Speaking about transitive frames in propositional languages

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Abstract

This is a report about comparative study of the propostional intutionistic (non-modal) and classical modal languages interpreted in the standard way in transitive frames. Talking about transitive frames, the intutionisitc language displays some unusual features: its expressive power becomes weaker than that of the modal language, the induced consequence relation does not have a deduction theorem and etc. We develops a manageable model theory for this consequence relation and its extensions which also reveals some unexpected phenomena. The balance between the intuitionistic and modal language is restore by adding to the former one more implication. This report is an extended abstract of [7].

1. Both modal and intuitionistic propositional languages may be regarded as talking about *quasi-order* $\mathcal{F} = \langle W, R\rangle$, R a reflexive and transitive relation on a set W. The primitive operators of the modal language $\mathcal{ML}$ are $\wedge$, $\vee$, $\rightarrow$, $\bot$ and $\Box$. The primitive operators of the intuitionisitic language $\mathcal{L}$ are same to the modal language without $\Box$. They are interpreted on quasi-order in usual way. For instance, $\Box$, $\to$ of $\mathcal{ML}$ and $\to$ of $\mathcal{L}$ are defined as follows if we denote truth-relation as $\models$;

the case of $\mathcal{ML}$

$$x \models \Box\varphi \quad \textit{iff} \quad \forall y \in W.(xRy \Rightarrow y \models \varphi)$$

$$x \models \varphi \to \psi \quad \textit{iff} \quad x \models \varphi \text{ implies } x \models \varphi$$

the case of $\mathcal{L}$

$$x \models \varphi \to \psi \quad \textit{iff} \quad \forall y \in W.(xRy \wedge y \models \varphi \Rightarrow y \models \psi)$$

The intuitionistic lanugage $\mathcal{L}$ may be evaluated on the set $\mathrm{Up}W = \{X \subseteq W : \forall x, y(x \in X \wedge xRy \Rightarrow y \in X)\}$ of *cones* (or *upward closed sets*). That means, for any intuitionisitic formulas, if the truth-sets $\mathcal{V}(\varphi)$ is defined as the set $\{x \in W : x \models \varphi\}$, $\mathcal{V}(\varphi) \in \mathrm{Up}W$ holds. Intuitionistic formulas cannot distinguish between points in the same cluster $C(x) = \{x\} \cup \{y \in W : xRy \wedge yRx\}$, however,as far as only cones are concerned, the modal and intuitionistic languages are of the same expressive power at both functional (local) and axiomatic (global) levels.

Let fix a quasi-order $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, and suppose $\mathcal{V}$ is a valuation on $\mathcal{F}$ and $\varphi(p_1, \ldots, p_n)$ is any $\mathcal{ML}$- and $\mathcal{L}$-formula where variables occuring in the list p_1, ..., p_n. We define a n-ary operator $\widehat{\varphi}_{\mathcal{F}}(X_1, \ldots, X_n)$ as the function $\widehat{\varphi}_{\mathcal{F}}(\mathcal{V}(p_1), \ldots, \mathcal{V}(p_n)) = \mathcal{V}(\varphi)$. $\varphi_{\mathcal{F}}$ is equal to $\widehat{\varphi}_{\mathcal{F}}$ if φ is $\mathcal{L}$-formula.

If φ is $\mathcal{ML}$-formula,

$$\varphi_{\mathcal{F}} = \begin{cases} \widehat{\varphi}_{\mathcal{F}} & \text{if } \widehat{\varphi}_{\mathcal{F}}(X_1, \ldots, X_n) \in \mathrm{Up}W \text{ for all } X_1, \ldots, X_n \in \mathrm{Up}W \\ \widehat{\bot}_{\mathcal{F}} & \text{otherwise.} \end{cases}$$

Proposition 1 *For any quasi-order $\mathcal{F}$, $\{\varphi_{\mathcal{F}} : \varphi \in \mathcal{L}\} = \{\varphi_{\mathcal{F}} : \varphi \in \mathcal{ML}\}$.*

Proof One direction ($\subseteq$) is easy by using *Gödel translation.* See [4]. For the converse direction, see Lemmas 8.32 and 8.33 in [4]. ❑

A class $\mathcal{C}$ of quasi-order is said to be $\mathcal{L}$- (or $\mathcal{ML}$-) *axiomatic* if there is a set Γ of $\mathcal{L}$- (respectively, $\mathcal{ML}$-) formulas such that, for every quasi-order $\mathcal{F}$, $\mathcal{F} \models \Gamma$ iff $\mathcal{F} \in \mathcal{C}$. ($\mathcal{F} \models \Gamma$ means that all formulas in Γ are true at all points in $\mathcal{F}$ under all possible valuations.) Since $\mathcal{L}$-formulas do not distinguish between points in one cluster, when comparing the axiomatic power of modal and intuitionistic formulas we should consider skelton-closed frame classes. Here, a class of frames is *skelton-closed* if with every $\mathcal{F}$ contains also all the quasi-order whose skeltons are isomprphic to the skelton of $\mathcal{F}$. We define a *skelton* of quasi-order $\mathcal{F} = \langle W, R\rangle$ as $\mathcal{F}^\circ = \langle W^\circ, R^\circ\rangle$ where $W^\circ = \{C(x) : x \in W\}$ and $C(x)R^\circ C(y)$ iff xRy.

Proposition 2 *A skeleton-closed class $\mathcal{C}$ of quasi-orders is $\mathcal{L}$-axiomatic iff it is $\mathcal{ML}$-axiomatic.*

Proof The deduction from the assumpotion a class is $\mathcal{ML}-$ *axiomatic* to it is $\mathcal{L}$-axiomatic, see [11]. The converse direction is easy. ❑

Example 3 *The class of all partial orders without infinite strictly ascending chains is $\mathcal{ML}$-axiomatic; it is axiomatizable by the Grzegorczyk formula* $\Box(\Box(p \to \Box p) \to p) \to p$ *but not $\mathcal{L}$-axiomatic; it is not skeleton-closed.*

Between $\mathcal{ML}$ and $\mathcal{L}$, there is te fact that the Gödel translation $\mathbf{T}$ embeds extensions of intuitionisitic logic **Int** into extensions of classical modal logic **S4**. We denote the class of extensions of **Int** known as *super-intuitionistic* or *intermediate logics* (si-logics, for short) as Ext**Int**, and smallest si-logic containig a set of $\mathcal{L}$-formulas Γ as $\mathbf{Int} + \Gamma$. Each si-logic contains **Int**, and is closed under modus ponens (MP) and substituition (Subst). NExt**S4** is the class of normal extensions of **S4** which are sets of $\mathcal{ML}$-formulas containing **S4** and closed under Subst, MP and necessitation. $\mathbf{S4} \oplus \Gamma$ is the smallest normal extension of **S4** to contain $\Gamma \subseteq \mathcal{ML}$.

Define a map ρ : Next**S4** $\mapsto$ Ext**Int** and τ, σ : Ext**Int** $\mapsto$ Next**S4** by taking, for any $M \in$ NExt**S4** and $L \in$ Ext**Int**,

$$\begin{aligned} \rho M &= \{\varphi \in \mathcal{L} : \mathbf{T}\varphi \in M\}, \\ \tau L &= \mathbf{S4} \oplus \{\mathbf{T} : \varphi \in L\}, \\ \sigma L &= \tau L \oplus \Box(\Box(p \to \Box p) \to p) \to p. \end{aligned}$$

(Detailed properties the above mapings, for instance, see [4].)

$\Box$ of **S4** can be understand as denoting informal provability. As contrasted with **S4**, the fact $\Box$ of **GL** denotes formal provability in Peano arithmetic is well known. Using $\mathbf{T}^{+}$, we can get embedding of **Int** into **GL**, where $\mathbf{T}^{+}(\varphi)$ replaces every $\Box\psi$ in $\mathbf{T}(\varphi)$ by $\Box^{+}\varphi = \varphi \wedge \Box\varphi$. Visser [8] described ρ**GL** and ρ**K4** (**K4** is the modal logic of all transitive frames) in the form of natural deduction systems.

Ruitenburg [6], criticizing the BHK interpretation of **Int** for not explaining the logical connectives in simpler terms, proposed to interpret implication as "a proof of $\varphi \rightarrow \psi$ is a construction that uses the assumption φ to produce a proof of ψ". And he shows that his proof interpretation gives rise not to **Int** but a weaker logic which is characterized by the class of arbitrary transitive (not necessary reflexive) frames.

Our aims are to clarify how far the relation ship between $\mathcal{L}$ and $\mathcal{ML}$ considered above can be extended on the class of frames which relation is traisitive, and to find a suitable non-modal propositional language which could talk about transitive frames as fulently as $\mathcal{L}$ can talk about quasi-orders.

2. From now on by a *(Kripke) frame* we mean a pari $\mathcal{F} = \langle W, R \rangle$ in which R is a transitive relation on a set $W \neq \emptyset$.A *model* of the language $\mathcal{L}$ is a pair $\mathcal{M} = \langle \mathcal{F}, \mathcal{V} \rangle$, where $\mathcal{F}$ is a frame and $\mathcal{V}$ maps propositional variables into UpW. The truth-relation $\models$ in $\mathcal{M}$ is same to the case for **Int**. $\mathcal{M} \models \varphi$, φ is *true* in $\mathcal{M}$, means that $x \models \varphi$ for every $x \in W$, and $\mathcal{F} \models \varphi$, φ is *valid* in $\mathcal{F}$, that φ is true in every model on $\mathcal{F}$.

From this definition, $\top \rightarrow \bot$ (where $\top$ is $\bot \rightarrow \bot$) is true at every final irreflexive point in any model holds. As for the expressive power, the following propositions hold.

Proposition 4 *For all frame $\mathcal{F}$, $\{\varphi_{\mathcal{F}} : \varphi \in \mathcal{L}\} \subset \{\varphi_{\mathcal{F}} : \varphi \in \mathcal{ML}\}$.*

Proof Suppose translation $\mathbf{T}'$ which prefixes $\Box$ to every subformula of *varphi* of the form $\psi \to \chi$. To show proper inclusion holds, consider the frame $\mathcal{F} = \langle\{a, b\}, \emptyset\rangle$ and $\Box^{+}\neg p$. $\Box$

Proposition 5 *The class $\mathcal{Q}$ of all quasi-orders is $\mathcal{ML}$-axiomatic but not $\mathcal{L}$-axiomatic.*

Proof $\mathcal{F} \in \mathcal{Q}$ iff $\mathcal{F} \models \Box p \to p$. On the other hand, every $\mathcal{L}$-formula $\varphi \in$ **Int** (and even $\varphi \in$ **Cl**) is valid also in the frame $\langle\{a\}, \emptyset\rangle$, as is easily shown by induction on the construction of φ. So if $\mathcal{Q}$ would be axiomatizable by a set of $\mathcal{L}$-formulas Γ then $\Gamma \subseteq$ **Int** and consequently $\langle\{a\}, \emptyset\rangle \in \mathcal{Q}$, which is a contradiction. $\Box$

Let us consider now the set $\mathbf{V} = \{\varphi \in \mathcal{L} : \forall\mathcal{F}\ \mathcal{F} \models \varphi\}$. According to the completeness theorem of Visser [8], $\mathbf{V}$ coincides with the set of formulas derivable in the basic propositional logic BPL represented by Visser in the form of a natural deduction system.

To compare $\mathbf{V}$ with the standard axiomatization of **Int**, we just cite here the following observation from [8].

Proposition 6 $\mathbf{V}$ *is closed under substitution and modus ponens, and contains all the axioms of* **Int** *in [4] except $(p \to (q \to r)) \to ((p \to q) \to$*

$(p \rightarrow r))$.

Semantically the consequence relation $\vdash_{\mathbf{Int}}$ in intuitionistic logic can be defined as "$\Gamma \vdash_{\mathbf{Int}}$ iff $\forall \mathcal{M} \forall x((\mathcal{M}, x) \models \Gamma \Rightarrow (\mathcal{M}, x) \models \varphi)$",where $\mathcal{M}$ ranges over intuitionistic models and x over points in $\mathcal{M}$. As was shown by Visser [8], the relation $\vdash_{\mathbf{V}}$ defined by "$\Gamma \vdash_{\mathbf{V}}$ iff $\forall \mathcal{M} \forall x((\mathcal{M}, x) \models \Gamma \Rightarrow (\mathcal{M}, x) \models \varphi)$",where $\mathcal{M}$ ranges over all transitive models, is the consequence relation of his natural deduction system for $\mathbf{V}$.

Now, considering $\langle \mathcal{L}, \vdash_{\mathbf{V}} \rangle$ as a deductive system, we see that modus ponens is nota derivable rule in it. Moreover,

Proposition 7 *There exists no formula $\chi(p, q)$ such that,for all Γ, φ, ψ,*

$$\Gamma, \psi \vdash_{\mathbf{V}} \varphi \textit{ iff } \Gamma \vdash_{\mathbf{V}} \chi(\psi, \varphi).$$

Proof Assume on contrary. Take Γ as $\{\top \rightarrow \bot\}$,and derive the contradiction. □

3. The Kripke semantics we considered in the previous section is not enough for dealing with extensions of $\mathbf{V}$. An algebraic sematics for $\mathbf{V}$ was introduced by Ardeshir and Ruitenburg [2]. The aim of this section is to define a notion of a general frame for $\mathbf{V}$ and develop to some extent duality theory for the algebraic and relational semantics.

We can get an impression how algebras for $\mathbf{V}$ may look like by representing transitive frames $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ as the algebras of cones $\mathcal{F}^+ =$

$\langle \mathrm{Up}W, \cap, \cup, \rightarrow, \emptyset, W\rangle$ in which

$$X \rightarrow Y \;=\; \{x \in W : \forall y\ (xRy \wedge y \in X \Rightarrow y \in Y)\} \tag{1}$$

(the logical connectives $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\top$ are interpreted in $\mathcal{F}^+$ by the operations $\cap$, $\cup$, $\rightarrow$, $\emptyset$, W, respectively). Every such algebra is clearly a bounded (i.e., with top and bottom) distributive lattice satisfying the following equations ($a \leq b$ means $a \wedge b = a$):

$$a \rightarrow b \wedge c = (a \rightarrow b) \wedge (a \rightarrow c);$$

$$b \vee c \rightarrow a = (b \rightarrow a) \wedge (c \rightarrow a);$$

$$a \rightarrow a = \top \text{ and } a \leq \top \rightarrow a;$$

$$(a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c.$$

Let us take these properties as a definition and call a bounded distributive lattice $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, \bot, \top\rangle$ satisfying the equations above a **V**-*algebra.* Our goal now is to show that all **V**-algebras are induced by frames, are subalgebras of the corresponding algebras of cones, to be more exact. To this end we require the following lemma on the existence of prime filters in **V**-algebras.

Lemma 8 *Suppose $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, \bot, \top\rangle$ is a* **V**-*algebra, ∇ a prime filter in $\mathcal{A}$ and let C and D be subsets of A such that*

$$\forall c_1, \ldots, c_m \in C\ \forall d_1, \ldots, d_n \in D\ c_1 \wedge \ldots \wedge c_m \rightarrow d_1 \vee \ldots \vee d_n \notin \nabla.$$

Then there exists a prime filter ∇' in $\mathcal{A}$ such that $C \subseteq \nabla'$, $\nabla' \cap D = \emptyset$

and $\nabla R \nabla'$, *where*

$$\nabla R \nabla' \text{ iff } \forall a, b \in A \ (a \to b \in \nabla \wedge a \in \nabla' \Rightarrow b \in \nabla').$$

Theorem 9 *All subalgebras of algebras of the form* $\mathcal{F}^+$, $\mathcal{F}$ *a transitive frame, comprise (up to isomorphism) the variety (equational class) of* $\mathbf{V}$*-algebras.*

Proof When we prove closeness of the operator $\to$ (defined by (1)), use lemma 8. ❑

Following the standard model-theoretic terminology of modal logic, we call a *general* $\mathbf{V}$*-frame* any structure $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}, \mathcal{P} \rangle$ where $\langle W, R \rangle$ is a Kripke frame and P a set of R-cones containing $\emptyset$ and closed under $\cap$, $\cup$ and $\to$ defined by (1). If $P = \mathrm{Up}W$, we call $\mathcal{F}$ a Kripke frame as before and may not mention P explicitly. The *dual* of $\mathcal{F}$, denoted by $\mathcal{F}^+$, is the subalgebra of $\langle W, R \rangle^+$ with domain P.

Theorem 10 *A general* $\mathbf{V}$*-frame* $\mathcal{F} = \langle W, R, P \rangle$ *is isomorphic to* $(\mathcal{F}^+)_+$ *iff* $\mathcal{F}$ *is descriptive in the sense that*

- $x = y$ *iff* $\forall X \in P \ (x \in X \Leftrightarrow y \in X)$;
- xRy *iff* $\forall X, Y \in P \ (x \in X \to Y \wedge y \in X \Rightarrow y \in Y)$;
- $\langle W, P \rangle$ *is compact, i.e., for all* $\mathcal{X} \subseteq P$ *and* $\mathcal{Y} \subseteq \{W - X : X \in P\}$, *if* $\mathcal{X} \cup \mathcal{Y}$ *has the finite intersection property then* $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

Proof Similar to the proof of Theorem 8.51 in [4]. ❑

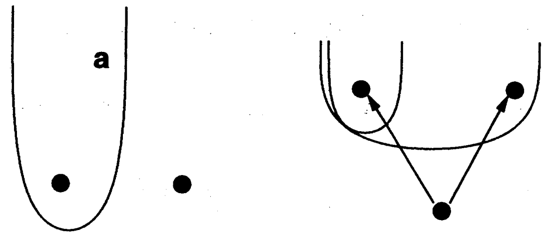

図 1: V-frames

Example 11 *Two examples of descriptive* **V**-*frames are shown in Fig. 1. The frame* $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}, \mathcal{P} \rangle$ *on the left consists of two irreflexive points (represented by* $\bullet$*) which do not see each other; all the cones in the set* P *of possible values, save* W *and* $\emptyset$, *are indicated explicitly by curve lines. Arrows in the second frame* $\mathcal{G}$ *define its accessibility relation. It may be of interest to notice that although these frames are finite, they are not Kripke frames (i.e., their sets of possible values do not contain all cones), which contrasts with the standard case of frames for modal and intuitionistic logics.*

Although **V** and $\vdash_{\mathbf{V}}$ are characterized by the variety of **V**-algebras, the connection between algebraic properties of this variety and the consequence relation $\vdash_{\mathbf{V}}$ is not as close as it is between, say, intuitionistic logic and Heyting algebras. For instance, almost all non-pathological propositional logics are protoalgebraic in the sense of Blok and Pigozzi [3]. However, as we show below, this is not the case for $\vdash_{\mathbf{V}}$.

Roughly speaking, a consequence relation $\vdash$ is protoalgebraic if there is a close connection between designated elements and congruences in matrices for $\vdash$. A syntactic definition looks like this. Say that two formulas α and β are Γ-*equivalent relative to* $\vdash$ if, for every formula γ and every variable p occurring on γ, "$\Gamma \vdash \gamma(\alpha/p)$ iff $\Gamma \vdash \gamma(\beta/p)$." Formulas α and β are Γ-*interderivable relative to* $\vdash$ if "$\Gamma, \alpha \vdash \beta$ iff $\Gamma, \beta \vdash \alpha$." Finally, $\vdash$ is called *protoalgebraic* if, for every set of formulas Γ, any two formulas are Γ-interderivable relative to $\vdash$ whenever they are Γ-equivalent relative to $\vdash$.

Theorem 12 $\vdash_{\mathrm{V}}$ *is not protoalgebraic.*

Proof We use the following algebraic characterization of protoalgebraic consequence relations. Consider a matrix $\mathsf{M} = (\mathcal{A}, D)$, i.e., an algebra $\mathcal{A}$ together with a subset D of the domain A of $\mathcal{A}$. M is a *matrix for* a consequence relation $\vdash$ if $\mathcal{V}(\Gamma) \subseteq D$ implies $\mathcal{V}(\varphi) \in \mathcal{D}$ whenever $\Gamma \vdash \varphi$ and $\mathcal{V}$ is a valuation in $\mathcal{A}$. By ΩD we denote the largest congruence relation in $\mathcal{A}$ which respects D, i.e., such that $(a, b) \in \Omega D$ implies $a \in D$ iff $b \in D$. Blok and Pigozzi [3] showed that a consequence relation $\vdash$ is protoalgebraic iff $D_1 \subseteq D_2$ implies $\Omega D_1 \subseteq \Omega D_2$ whenever $(\mathcal{A}, D_1)$ and $(\mathcal{A}, D_2)$ are matrices for $\vdash$. Consider now the matrices $(\mathcal{F}^+, \{\top\})$ and $(\mathcal{F}^+, \{\top, a\})$, where $\mathcal{F}$ is the frame defined in Example 11. Clearly, both of them are matrices for $\vdash_{\mathrm{V}}$. It is easily verified that $\Omega\{\top\}$ identifies only a and $\bot$ and $\Omega\{\top, a\}$ only $\top$ and a. Hence $\Omega\{\top\} \not\subseteq \Omega\{\top, a\}$, and so $\vdash_{\mathrm{V}}$ is not protoalgebraic. ❑

4. When we consider about the extensions of **V**, we encounter the problem what kind of extensions are worth considering is. As we observed in proposition 4, the class of quasi-order is not $\mathcal{L}$-axiomatic, we cannot introduce *formula extension*, like Ext**Int**, as a set of formulas L that contains **V** and is closed under Subst and $\vdash_{\mathbf{V}}$ (that means $\varphi \in L$ if $\Gamma \subseteq L$ and $\Gamma \vdash_{\mathbf{V}} \psi$).

So we consider that extensions not of the logic **V** but of the consequence relation $\vdash_{\mathbf{V}}$. The most general class of such extensions consists of arbitrary finitary (i.e., if $\Gamma \vdash \varphi$ then $\Delta \vdash \varphi$ for some finite $\Delta \subseteq \Gamma$) structural (i.e., closed under substitution) consequence relations containing $\vdash_{\mathbf{V}}$. Each of them can be looked at as the result of adding to $\vdash_{\mathbf{V}}$ a set Ξ of inference rules. Let $\vdash_{\mathbf{V}} + \Xi$ denote the smallest finitary structural consequence relation containing $\vdash_{\mathbf{V}}$ and respecting the rules in Ξ. For instance, $\vdash_{\mathbf{Int}} = \vdash_{\mathbf{V}} + \dfrac{p, p \to q}{q}$ and $\vdash_{\mathbf{FPC}} = \vdash_{\mathbf{V}} + \dfrac{(\top \to p) \to \top}{\top \to p}$. The tautologies that are deduced by $\vdash_{\mathbf{FPC}}$ coincide with $\rho\mathbf{GL}$ (**FPC** stands for "formal propositional calculus").

We say that a consequence relation $\vdash$ is a **V**-*consequence* if it is finitary and characterized by a class $\mathcal{FR}$ of general **V**-frames in the sense that $\vdash$ coincides with the relation $\models_{\mathcal{FR}}$ such that $\Gamma \models_{\mathcal{FR}} \varphi$ iff for any model $\mathcal{M}$ based on a frame in $\mathcal{FR}$ and any point x in $\mathcal{M}$, $(\mathcal{M}, x) \models \Gamma \Rightarrow (\mathcal{M}, x) \models \varphi$. The class $\{\mathcal{F} : \vdash \subseteq \models_{\mathcal{F}}\}$ of frames for $\vdash$ will be denoted by $\mathrm{Fr}\vdash$.

The corresponding notions for **V**-algebras can be defined as follows.

For a class $\mathcal{AL}$ of $\mathbf{V}$-algebras we write $\Gamma \models_{\mathcal{AL}} \varphi$ iff there exists a finite subset Γ' of Γ such that the equation $\bigwedge \Gamma' \leq \varphi$ is valid in all members of $\mathcal{AL}$. The class $\{\mathcal{A} : \vdash \subseteq \models_{\mathcal{A}}\}$ of algebras for $\vdash$ is denoted by $\mathrm{Alg} \vdash$. Then,

Theorem 13 (i) *A class of* $\mathbf{V}$*-algebras is of the form* $\mathrm{Alg} \vdash$ *for a* $\mathbf{V}$*-consequence* $\vdash$ *iff it is a subvariety of the variety of all* $\mathbf{V}$*-algebras.*

(ii) *A class of general* $\mathbf{V}$*-frames is of the form* $\mathrm{Fr} \vdash$ *for a* $\mathbf{V}$*-consequence* $\vdash$ *iff it is closed under generated subframes, reductions, disjoint unions and it as well as its complement are closed under the formation of biduals.*

Proof (i) Let $\mathcal{AL} = \mathrm{Alg} \vdash$, for a $\mathbf{V}$-consequence $\vdash$. Then $\mathcal{AL}$ is the class of $\mathbf{V}$-algebras defined by the equations $\{\bigwedge \Gamma \leq \varphi : \Gamma \vdash \varphi, \Gamma \text{ is finite}\}$ and so $\mathcal{AL}$ is a variety. Conversely, given a variety $\mathcal{AL}$ contained in the variety of $\mathbf{V}$-algebras, one can easily check that $\mathrm{Alg} \models_{\mathcal{AL}}$ coincides with $\mathcal{AL}$.

(ii) The closure conditions for classes of the form $\mathrm{Fr} \vdash$ are clear. Conversely, assume that $\mathcal{FR}$ is a class of general $\mathbf{V}$-frames closed under generated subframes, reductions, disjoint unions and it as well as its complement are closed under the formation of biduals. First we show that $\models_{\mathcal{AL}}$ is finitary. To this end suppose that $\Gamma' \not\models_{\mathcal{AL}} \varphi$, for every finite subset Γ' of a set of formulas Γ. Take for each such Γ' a frame $\mathcal{F} \in \mathcal{FR}$ refuting $\Gamma' \vdash \varphi$ and form the disjoint union $\mathcal{G}$ of all those $\mathcal{F}$. Then in view of the compactness of the descriptive frame $(\mathcal{G}^{+})_{+} \in \mathcal{F}$, it must refute

$\Gamma \vdash \varphi$. It follows that $\models_{\mathcal{FR}}$ is a **V**-consequence. So it remains to show that $\mathcal{FR} = \mathrm{Fr} \models_{\mathcal{FR}}$. But this is obtained from (i) by using the results on duality between general **V**-frames and **V**-algebras (see [5] for a similar argument). $\square$

The consequence relations as the extensions of **V** are *complete* in the sense that, for any *finite* set of formulas Γ and formula φ, if $\Gamma \not\vdash \varphi$ then there exists a Kripke frame $\mathcal{F} \in \mathrm{Fr} \vdash$ such that $\Gamma \not\models_{\mathcal{F}} \varphi$. In contrast with superintuitionisitic logics it is almost trivial to construct incomplete **V**-consequeces.

Proposition 14 (i) *The consequence relation $\models_{\mathcal{G}}$, where $\mathcal{G}$ is the frame defined in Example 11, is not complete.*

(ii) $\vdash_{\mathbf{V}} + (p \to q) \vee (q \to p)$ *is not complete.*

Proof Let $\varphi_1 = (p \to q) \vee ((p \to q) \to p)$, $\varphi_2 = (p \to q) \vee (q \to p)$. One can easily show that a Kripke frame validates φ_1 iff it validates φ_2 iff it is linear. However, $\mathcal{G}$ refutes φ_1 but validates φ_2. The claims of the proposition follow immediately. $\square$

The class of all **V**-consequences order by inclusion forms a complete lattice; we denote it by Ext $\vdash_{\mathbf{V}}$. There is an isomorphism between Ext**Int** and NExt**Grz** via σ. So we conjecture that there exist an isomorphism between Ext $\vdash_{\mathbf{V}}$ and NExt**L**, for some $L \in$ NExt**K4**. But,

Theorem 15 *The lattice of* **V**-*consequences conating* $\vdash_{\mathbf{FPC}}$ *is not iso-*

morphic to the lattice NExt**GL**.

5. From the semantical point of view, all the "peculiarities" of the language $\mathcal{L}$ interpretd on thransitive frames as wll as of the logic **V** and its extensions we observed in the thress previous sections are explained by the fact that being in an irrflexive world x; we can talk about x using $\wedge$ and $\vee$; $\rightarrow$ is for talking about successors x. A way of improving the expressiveness power of $\mathcal{L}$ is to add the following one more implication $\hookrightarrow$ to $\mathcal{L}$;

$$x \models \varphi \hookrightarrow \psi \text{ iff } \forall y \in W((x = y \vee xRy) \wedge y \models \varphi \Rightarrow y \models \psi).$$

The resulting "biarrow" language is denoted by $\mathcal{L}_2$. But instead of $\mathcal{L}_2$, we can consider the modal language $\mathcal{ML}_{\hookrightarrow}$ which results from $\mathcal{ML}$ by replacing $\rightarrow$ with $\hookrightarrow$. Because, using valuation $\mathcal{V}$ of propositional variables in UpW, $\rightarrow$ (and $\Box$) can be defined via $\hookrightarrow$ and $\Box$ (respectively, $\rightarrow$ and $\top$) as follows;

$$x \models \varphi \hookrightarrow \psi \quad \text{iff} \quad x \models \Box(\varphi \hookrightarrow \psi),$$

$$x \models \Box\varphi \quad \text{iff} \quad x \models \top \rightarrow \varphi.$$

Before showing about the expressive powers, we introduce a caluculs. Let **U** be the set of $\mathcal{ML}_{\hookrightarrow}$-formulas that are valid in all transitive frames and let "$\Gamma \vdash_{\mathbf{U}} \varphi$ iff $\forall \mathcal{M} \forall x\ ((\mathcal{M}, x) \models \Gamma \Rightarrow (\mathcal{M}, x) \models \varphi)$". Clearly, the deduction theorem holds for $\vdash_{\mathbf{U}}$ and $\hookrightarrow$ (that is, "$\Gamma, \varphi \vdash_{\mathbf{U}} \psi$ iff $\Gamma \vdash_{\mathbf{U}} \varphi \hookrightarrow \psi$"), and it is easy to check that $\vdash_{\mathbf{U}}$ is protoalgebraic.

U can be considered as a normal modal logic on the intuitionistic basis. This observation and completeness results of [10] provide a Hilbert-style axiomatization for **U** and $\vdash_{\mathbf{U}}$.

Theorem 16 *The calculus U in the language $\mathcal{ML}_{\hookrightarrow}$ with modus ponens and substitution as its inference rules and the axioms*

1. *those of* **Int**,
2. $\Box(p \hookrightarrow q) \hookrightarrow (\Box p \hookrightarrow \Box q)$, $\Box p \hookrightarrow \Box\Box p$, $p \hookrightarrow \Box p$,
3. $\Box p \hookrightarrow (q \vee (q \hookrightarrow p))$

is strongly complete with respect to the class of transitive frames, i.e., $\Gamma \vdash_U \varphi$ *iff* $\Gamma \vdash_{\mathbf{U}} \varphi$.

In the proof of theorem 16, to interpret $\mathcal{ML}_{\hookrightarrow}$ that axiomatized by U, we use a notion of IM-frames from [10]. That is, descriptive IM-frames $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}_{\hookrightarrow}, \mathcal{R}, \mathcal{P} \rangle$ is a strucutre such that $\langle W, R_{\hookrightarrow}, P \rangle$ is a descriptive (quasi-ordered) frame for **Int**($\hookrightarrow$ is interpreted via $R_{\hookrightarrow}$), P is closed under the standard $\Box$ interpreted via R, xRy iff $\forall X \in P(x \in \Box X \Rightarrow y \in X)$ and $R_{\hookrightarrow} \circ R \circ R_{\hookrightarrow} = R$.

Remark. Not every general frame for **V** can be regarded as an IM-frame because it is not necessaarily closed under $\hookrightarrow$. So, IM-frames for **U** defined in the above will be called **U**-*frames*. Since $R_{\hookrightarrow}$ is uniquely determined R, we may omit $R_{\hookrightarrow}$ and denote these frames by $\mathcal{F} = \langle W, R, P \rangle$.

Now, NExt**U** of normal extensions of **U**, that is sets of $\mathcal{ML}_{\hookrightarrow}$-formulas containing **U** and closed under MP and Subst (the closure under necessitation is ensured by the axiom $p \hookrightarrow \Box p$). Then, immediately,

Theorem 17 *Every logic in* NExt**U** *is characterized by a class of (descriptive)* **U***-frames. Conversely, everyl class of general* **U***-frames determines a logic in* NExt**U**.

Using the result on embeddings of intuitionisitic modal logics into classical polymodal logics obtained in [9], [10], we can show that there is a relationship between NExt**U** and NExt**K4** is similar to that between Ext**Int** and NExt**S4** discussed in Section 1.

Let $\mathcal{ML}_2$ be the language with two necessity operators $\Box_I$ and $\Box$ (and the implication $\rightarrow$), and let T'' be the translation from $\mathcal{ML}_{\hookrightarrow}$ into $\mathcal{ML}_2$ prefixing $\Box_I$ to all subformulas and replacing $\hookrightarrow$ with $\rightarrow$. Given logics L_1 and L_2 in the unimodal languages $\mathcal{ML}_2 - \Box$ and $\mathcal{ML}_2 - \Box_I$, respectively, denote by $L_1 \otimes L_2$ their *fusion*, the smallest bimodal logic in $\mathcal{ML}_2$ to contain $L_1 \cup L_2$. By **IntK** we mean the minimal normal intuitionistic modal logic in the language $\mathcal{ML}_{\hookrightarrow}$ (i.e., the smallest set of formulas containing **Int**, the modal axiom of **K** and closed under modus ponens, substitution and necessitation). As is shown in [9],

(i) the map

$$\rho M = \{\varphi \in \mathcal{ML}_{\hookrightarrow} : \mathbf{T}''(\varphi) \in M\},$$

is a lattice homomorphism from NExt($\mathbf{S4} \otimes \mathbf{K}$) onto NExt**IntK** (preserving the finite model property and decidability);

(ii) each logic $\mathbf{IntK} \oplus \Gamma$ is embedded by $\mathbf{T}''$ into any logic M in the interval

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus \mathbf{T}''(\Gamma) \subseteq M \subseteq (\mathbf{Grz} \otimes \mathbf{K}) \oplus \mathbf{mix} \oplus \mathbf{T}''(\Gamma),$$

where $\mathbf{mix} = \Box_I \Box \Box_I p \leftrightarrow \Box p$, and

(iii) the map

$$\sigma(\mathbf{IntK} \oplus \Gamma) = (\mathbf{Grz} \otimes \mathbf{K}) \oplus \mathbf{mix} \oplus \mathbf{T}''(\Gamma)$$

is a lattice isomorphism from NExt**IntK** onto NExt($\mathbf{Grz} \otimes \mathbf{K}$) $\oplus$ **mix**. (As before, the operation $\oplus$ means "take the union and close it under the postulated inference rules".)

If we consider now **K4** as a bimodal logic in $\mathcal{ML}_2$ by defining $\Box_I \varphi = \varphi \wedge \Box\varphi$, then we may assume **K4** to be in the class NExt($\mathbf{S4} \otimes \mathbf{K4}$). Since this "bimodal" **K4** is characterized by the class of frames of the form $\langle W, R^r, R\rangle$ and in view of Proposition 21 in [10], $\rho\mathbf{K4} = \mathbf{U}$. Therefore, **U** has the finite model property and ρ is a lattice homomorphism from NExt**K4** onto NExt**U**. The logic

$$\mathbf{Grz}' = \mathbf{K4} \oplus \Box(\Box(p \to \Box p) \to p) \to \Box p$$

is known to be determined by the class of finite Kripke frames without proper (i.e., containing ≥ 2 points) clusters (see e.g. [1]). **U** is characterized by this class too. It follows that $\rho\mathbf{Grz}'$ is also **U**. And since

mix $\in$ **K4** and the "bimodal" **Grz**$'$ is in NExt(**Grz** $\otimes$ **K4**), we finally obtain

Theorem 18 *The map* σ *is an isomorphism from* NExt**U** *onto* NExt**Grz**$'$.

It is not hard to see also that modulo clusters the languages $\mathcal{ML}_{\hookrightarrow}$ and $\mathcal{ML}$ have the same functional power on the class of transitive frames.

Proposition 19 $\{\varphi_{\mathcal{F}} : \varphi \in \mathcal{ML}_{\hookrightarrow}\} = \{\varphi_{\mathcal{F}} : \varphi \in \mathcal{ML}\}$, *where* $\mathcal{F}$ *ranges over the class of all transitive frames.*

Proof Similar to the proof of Proposition 1. ❏

To prove that the languages under consideration have the same axiomatic power we require frame-based $\mathcal{ML}_{\hookrightarrow}$-formulas simulating canonical formulas for **K4** of [11]. Namely, with every finite rooted transitive frame $\mathcal{F} = \langle W, R\rangle$ without proper clusters—let $a_0, \dots, a_n$ be all its points and a_0 the root—and a set $\mathcal{D}$ of antichains in $\mathcal{F}$ we associate a formula $\gamma(\mathcal{F}, \mathcal{D}, \bot)$ which is the implication ($\hookrightarrow$) whose consequent is p_0 and the antecedent is the conjunction of all formulas of the form

$$
\begin{array}{rcll}
 & & \Box p_0 & \text{if } \neg a_0 R a_0, \\
 & & \Box p_i \hookrightarrow p_i & \text{if } a_i R a_i, \\
\gamma_{ij} & = & (\bigwedge \Gamma_j \hookrightarrow p_j) \hookrightarrow p_i & \text{if } a_i R a_j, \\
\gamma_{\mathrm{d}} & = & \bigwedge\limits_{a_j \in W - \lceil \uparrow} (\bigwedge \Gamma_j \hookrightarrow p_j) \hookrightarrow \bigvee\limits_{a_i \in \mathrm{d}} p_i & \text{if } \mathrm{d} \in \mathcal{D}, \\
\gamma_{\bot} & = & \bigwedge\limits_{i=0}^{n} (\bigwedge \Gamma_j \hookrightarrow p_j) \hookrightarrow \bot, &
\end{array}
$$

where

$$\Gamma_j = \begin{cases} \{p_k : a_k \notin a_j\underline{\uparrow}\} & \text{if } a_jRa_j \\ \{\Box p_j, p_k : a_k \notin a_j\underline{\uparrow}\} & \text{if } \neg a_jRa_j, \end{cases}$$

and

$$X\uparrow = \{y \in W : \exists x \in X\ xRy\},\ X\underline{\uparrow} = X \cup X\uparrow,$$

$$X\downarrow = \{y \in W : \exists x \in X\ yRx\},\ X\overline{\downarrow} = X \cup X\downarrow .$$

Given a frame $\mathcal{G} = \langle V, S\rangle$, a partial map f from V onto W is called a *subreduction* of $\mathcal{G}$ to $\mathcal{F}$ if, for all $x, y \in \mathrm{dom}f$,

(R1) xSy implies $f(x)Rf(y)$;

(R2) $f(x)Rf(y)$ implies $\exists z \in x\uparrow\ f(z) = f(y)$.

A subreduction f is said to be *cofinal* if $\mathrm{dom}f\uparrow \subseteq \mathrm{dom}f\overline{\downarrow}$.

Proposition 20 *For any transitive frame $\mathcal{G} = \langle \mathcal{V}, \mathcal{S}\rangle$, $\mathcal{G} \not\models \gamma(\mathcal{F}, \mathcal{D}, \bot)$ iff there is a cofinal subreduction of $\mathcal{G}$ to $\mathcal{F}$ satisfying the following (closed domain) condition*

(CDC) $\neg\exists x \in \mathrm{dom}f\uparrow - \mathrm{dom}f\ \exists \mathrm{d} \in \mathcal{D}\ f(x\uparrow) = \mathrm{d}\underline{\uparrow}$.

Proof ($\Rightarrow$) Suppose $\mathcal{G}$ refutes $\gamma(\mathcal{F}, \mathcal{D}, \bot)$ under some valuation (in $\mathrm{Up}V$) and π is the premise of $\gamma(\mathcal{F}, \mathcal{D}, \bot)$. Define a partial map from V onto W by taking, for $x \in V$,

$$f(x) = \begin{cases} a_i & \text{if } x \not\models p_i,\ x \models \Gamma_i,\ x \models \pi \\ \text{undefined} & \text{otherwise} \end{cases}$$

and show that it is a cofinal subreduction of $\mathcal{G}$ to $\mathcal{F}$ satisfying (CDC). Notice first that f is a partial function. Indeed, since $\mathcal{F}$ contains no proper clusters, if $a_i \neq a_j$ then either $\neg a_i R a_j$ or $\neg a_j R a_i$; in the former case $p_j \in \Gamma_i$ and in the latter $p_i \in \Gamma_j$.

Let xSy, $f(x) = a_i$ and $f(y) = a_j$. Then (since the valuation is intuitionistic) $x \not\models p_j$ from which $p_j \notin \Gamma_i$ and so $a_j \in a_i\underline{\uparrow}$, i.e., either $a_i R a_j$ or $a_i = a_j$. Now, if $a_i = a_j$ and $\neg a_i R a_i$ then $\Box p_i \in \Gamma_i$, so $x \models \Box p_i$ and $y \models p_i$, which is a contradiction. Thus, f satisfies (R1). To show that it satisfies (R2) suppose $f(x) = a_i$ and $a_i R a_j$. If $a_i \neq a_j$ then $x \not\models p_i$, $x \models \gamma_{ij}$, and so there is $y \in x\uparrow$ such that $y \models \Gamma_j$ and $y \not\models p_j$, i.e., $f(y) = a_j$. And if $a_i = a_j$ then, since $x \not\models p_i$ and $x \models \Box p_i \hookrightarrow p_i$, we have $x \not\models \Box p_i$, i.e., there is $y \in x\uparrow$ such that $y \not\models p_i$, and again $f(y) = a_i$.

Since, by the definition, $f(x) = a_0$ whenever $x \not\models \gamma(\mathcal{F}, \mathcal{D}, \bot)$, the map f is a surjection. The fact that f is cofinal is clearly ensured by the conjunct $\gamma_\bot$ and that it satisfies (CDC) by γ_Γ.

($\Leftarrow$) Let f be a cofinal subreduction of $\mathcal{G}$ to $\mathcal{F}$ satisfying (CDC). Define a valuation in $\mathcal{G}$ by taking

$$x \models p_i \text{ iff } x \notin f^{-1}(a_i)\overline{\downarrow}.$$

By a straightforward inspection one can easily verify that under this valuation $x \not\models \gamma(\mathcal{F}, \mathcal{D}, \bot)$ for every $x \in f^{-1}(a_0)$. ❑

Corollary 21 *For every Kripke frame $\mathcal{G}$, every finite rooted frame $\mathcal{F}$*

without proper clusters and every set $\mathcal{D}$ of antichains in $\mathcal{F}$,

$$\mathcal{G} \not\models \alpha(\mathcal{F}, \mathcal{D}, \bot) \text{ iff } \mathcal{G} \not\models \gamma(\mathcal{F}, \mathcal{D}, \bot).$$

Proof Follows from Proposition 20 and the refutability criterion for canonical formulas in [11]. ❑

Remark. Actually, it is not hard to show that Proposition 20 holds for any general **U**-frame $\mathcal{G}$. It follows that the formulas of the form $\gamma(\mathcal{F}, \mathcal{G}, \bot)$ are enough to axiomatize all logics in NExt**U**.

Proposition 22 *A skeleton-closed class $\mathcal{C}$ of transitive frames is $\mathcal{ML}_{\hookrightarrow}$-axiomatic iff it is $\mathcal{ML}$-axiomatic.*

Proof If $\mathcal{C}$ is axiomatized by a set Γ of $\mathcal{ML}_{\hookrightarrow}$-formulas then it is also axiomatizable by the set $\mathbf{T}''(\Gamma)$. Suppose now that L is the logic in $\mathcal{ML}$ characterized by $\mathcal{C}$. Since $\mathcal{C}$ is skeleton-closed, it is axiomatizable by a set Γ of canonical formulas for **K4** built on frames without proper clusters. The logic $\rho L \in$ NExt**U** is also characterized by $\mathcal{C}$. It follows that $\gamma(\mathcal{F}, \mathcal{D}, \bot) \in \rho\mathcal{L}$ whenever $\alpha(\mathcal{F}, \mathcal{D}, \bot) \in \Gamma$. Now, if $\mathcal{G} \notin \mathcal{C}$ then $\mathcal{G} \not\models \alpha(\mathcal{F}, \mathcal{D}, \bot)$, for some $\alpha(\mathcal{F}, \mathcal{D}, \bot) \in \Gamma$ and so $\mathcal{G} \not\models \gamma(\mathcal{F}, \mathcal{D}, \bot)$. Thus, $\mathcal{C}$ is axiomatized by ρL (or by the $\mathcal{ML}_{\hookrightarrow}$-formulas $\gamma(\mathcal{F}, \mathcal{D}, \bot)$ such that $\alpha(\mathcal{F}, \mathcal{D}, \bot) \in \Gamma$). ❑

As we saw in Section 2, not all $\mathcal{ML}$-definable skeleton-closed classes of transitive frames are $\mathcal{L}$-definable. The situation changes drastically, however, when we consider frame classes definable by rules. Call a class of general **V**-frames *$\mathcal{L}$-rule definable* if it is of the form Fr $\vdash$, for some **V**-consequence $\vdash$. A class of transitive Kripke frames is *$\mathcal{L}$-rule definable* if it coincides with the subclass of all Kripke frames in some $\mathcal{L}$-rule definable class of general **V**-frames.

Theorem 23 *(i) Let $\mathcal{C}$ be an $\mathcal{L}_2$-definable class of general* **U***-frames. Then there exists an $\mathcal{L}$-rule definable class $\mathcal{C}'$ of general* **V***-frames such that $\mathcal{C}$ coincides with the subclass of all* **U***-frames in $\mathcal{C}'$.*

(ii) A class of Kripke frames is $\mathcal{L}_2$-definable iff it is $\mathcal{L}$-rule definable.

Proof Clearly, (ii) follows from (i), and to prove (i) it suffices to show that for any $\mathcal{L}_2$-definable class of descriptive **U**-frames, there exists an $\mathcal{L}$-rule definable class $\mathcal{C}'$ of descriptive **V**-frames such that $\mathcal{C}$ consists of precisely the **U**-frames in $\mathcal{C}'$ (for a **V**-frame $\mathcal{F}$ is a **U**-frame iff $(\mathcal{F}^+)_+$ is a **U**-frame). To this end consider the variety $\mathcal{V}$ of **V**-algebras generated by $\mathcal{C}^+ = \{\mathcal{F}^+ : \mathcal{F} \in \mathcal{C}\}$. $\mathcal{V} = HSP\mathcal{C}^+$, where H denotes the operation of taking homomorphic images, S the operation of taking subalgebras, and P the operation of forming direct products. It is enough to show that for any $\mathcal{A} \in \mathcal{V}$ such that $\mathcal{A}_+$ is a **U**-frame, we have $\mathcal{A}_+ \in \mathcal{C}$. Suppose that $\mathcal{A} \in HSP\mathcal{C}^+$ and $\mathcal{A}_+$ is a **U**-frame. Then $\mathcal{A} \in HS\mathcal{C}^+$, since $\mathcal{C}^+$ is closed under products. By the fact that there are descriptive frames $\mathcal{H}$ and $\mathcal{G}$ such that $\mathcal{G} \in \mathcal{C}$, $\mathcal{A}_+$ is a generated subframe of $\mathcal{H}$ and $\mathcal{G}$ is reducible to $\mathcal{H}$ by some f. For a frame $\mathcal{F} = \langle W, R, P \rangle$, denote by P^b the smallest set of cones containing P and such that $\mathcal{F}^b = \langle W, R, P^b \rangle$ is a **U**-frame. In other words, P^b is the closure of P under the operations $\hookrightarrow$, $\rightarrow$, $\cap$ and $\cup$. One can easily show that $\mathcal{H}^b$ is a reduct of $\mathcal{G}^b = \mathcal{G}$ (since $f^{-1}(X \odot Y) = f^{-1}(X) \odot f^{-1}(Y)$, for $\odot \in \{\hookrightarrow, \rightarrow, \cap, \cup\}$) and that $\mathcal{A}_+ = (\mathcal{A}_+)^b$ is a generated subframe of $\mathcal{H}^b$. And since $\mathcal{C}$ is closed under generated subframes, which are **U**-frames, and reducts, which are also **U**-frames, we finally obtain $\mathcal{A}_+ \in \mathcal{C}$. ❑

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